Elementary Transformations

X sm. surface, E v.b. of rk e on X, $C \subseteq X$ a sm. curve.
Let A be a line bundle on C of degree d s.t.

there is a surjection $E|_C \to A$.

Let $\mathcal{A}$ be A viewed as a torsion sheaf on X (extend by 0)

Then $E \twoheadrightarrow E|_C \twoheadrightarrow \mathcal{A}$. Let $F = \ker(E \to \mathcal{A})$.

so we have $0 \to F \xrightarrow{\mu} E \to \mathcal{A} \to 0$.

<u>Claim</u>: F is locally free. (F is called the <u>elementary transformation</u> of E corr. to $E \to \mathcal{A}$)

<u>Pf</u>: Take the SES on stalks at $x \in C$.

$$0 \to F_x \to E_x \to \mathcal{A}_x \to 0$$

Tensoring by $\mathcal{O}_x$ (str. sheaf of x):

$$\underset{\substack{\| \\ 0 \\ E_x \text{ free}}}{\mathrm{Tor}^1(E_x, \mathcal{O}_x)} \to \mathrm{Tor}^1(\mathcal{A}_x, \mathcal{O}_x) \to F_x \otimes \mathcal{O}_x \to E_x \otimes \mathcal{O}_x \to \mathcal{A}_x \otimes \mathcal{O}_x \to 0$$

If R is local ring at x, we have

$$0 \to R \xrightarrow{f} R \to A_x \to 0 \quad \text{free resolution.}$$

so $\mathrm{Tor}^1(\mathcal{A}_x, \mathcal{O}_x) \cong \mathcal{O}_x$

$\Rightarrow \dim(F_x \otimes \mathcal{O}_x) = \dim(E_x \otimes \mathcal{O}_x) = e$ $\square$

<u>Chern classes of F:</u>

Splitting principle $\Rightarrow c_1(\det F) = c_1(F)$

Taking Λ^e, get $0 \to \det(F) \to \det(E) \to \det(E) \otimes \mathcal{O}_C \to 0$

$\det(F) = \det(E)(-C) \Rightarrow c_1(F) = c_1(E) - [C]$.

$c_2(F)$? $\quad c(E) = c(F)c(\mathcal{A})$

$\Rightarrow c_2(E) = c_1(F)c_1(\mathcal{A}) + c_2(F) + c_2(\mathcal{A})$

$\Rightarrow c_2(F) = c_2(E) - c_1(E)\cdot[C] + [C]^2 - c_2(\mathcal{A})$

<u>Exercise:</u> Finish calculation.

Dualizing, get injection $0 \to E^* \xrightarrow{\mu^*} F^* \to \mathcal{B} \to 0$, w/ $\mathcal{B}$ supp. on C.

<u>Claim:</u> $\mathcal{B}$ is ext. by 0 of l.b. $B = N_{C/X} \otimes A^*$.

<u>Pf:</u> $0 \to F \to E \to \mathcal{A} \to 0$

$$0 \to \underset{\substack{\| \\ 0}}{\mathcal{H}om(\mathcal{A}, \mathcal{O}_X)} \to E^* \to F^* \to \mathcal{E}xt^1(\mathcal{A}, \mathcal{O}_X) \to 0$$

So $\mathcal{B} = \mathcal{E}xt^1(\mathcal{A}, \mathcal{O}_X)$

First assume $\mathcal{A} = \mathcal{O}_C$.

Then we have $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$

$\Rightarrow$ $0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{E}xt^1(\mathcal{O}_C, \mathcal{O}_X) \to 0$

$\Rightarrow$ $B = \mathcal{O}_C(C)$, as desired.

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Now we show claim holds for $A \Leftrightarrow$ holds for $A(P)$
$P \in C$.

Let $\mathcal{A}' = A(P)$ ext. by 0 to X, B', $\mathcal{B}'$ defined analogously.

Then $0 \to \mathcal{A} \to \mathcal{A}' \to \mathcal{O}_P \to 0$

$$0 \to \underset{\overset{\|}{0}}{\mathcal{E}xt^1(\mathcal{O}_P, \mathcal{O}_X)} \to \mathcal{E}xt^1(\mathcal{A}', \mathcal{O}_X) \to \mathcal{E}xt^1(\mathcal{A}, \mathcal{O}_X)$$
$$\to \mathcal{E}xt^2(\mathcal{O}_P, \mathcal{O}_X) \to 0$$

(higher $\mathcal{E}xt$s vanish since $\mathcal{A}'$ is resolved in two steps.)

<u>Exercise</u>: $\mathcal{E}xt^2(\mathcal{O}_P, \mathcal{O}_X) \cong \mathcal{O}_P$, $\mathcal{E}xt^1(\mathcal{O}_P, \mathcal{O}_X) = 0$

So $B = \mathcal{O}_C(C) \otimes A^* \Longleftrightarrow B' = \mathcal{O}(C) \otimes A^*(-P)$ $\square$